The q-state Potts-glass on d-dimensional hypercubic lattices

1994 J. Phys. A: Math. Gen. 27 1079

(http://iopscience.iop.org/0305-4470/27/4/005)

J. Phys. A: Math. Gen. 27 (1994) 1079–1097. Printed in the UK

The q-state Potts-glass on d-dimensional hypercubic lattices

G Schreider and J D Reger

Johannes Gutenberg Universität Mainz, Institut für Physik, D-55099 Mainz, Federal Republic of Germany

Received 26 July 1993

Abstract. We use a high-temperature star-graph expansion to compute the free energy and the susceptibility of a q-state Potts-glass with bimodal distribution of the random bonds. The series are up to order O(20) in $K = \beta J$ for arbitrary number of states q on d-dimensional simple cubic lattices. The result for the three-state Potts-glass in three dimensions shows excellent consistency with existing Monte Carlo data.

1. Introduction

The Potts-glass is assumed to be a generic model for realistic glasses, which do not possess the spin-inversion symmetry of Ising spin glasses [1,2]. Especially the first non-trivial realization, i.e. the $q = 3$ case has been studied by Monte Carlo (MC) methods [3]. There is a continuing interest in the question of whether the lower critical dimension of the short-range three-state Potts-glass is $d_c = 3$ or not. A high temperature series expansion for the three-state Potts model was presented for dimensions $d = 2$–8 [4]. From the behaviour of the correlation length it was suggested that the lower critical dimension is $d_l = 3$ for the $q = 3$ Potts-glass The Potts-glass in higher dimensions was investigated by renormalization-group studies [5]and replica mean-field incorporating fluctuations [6]The question about the upper critical dimension also seems to be an open one. We study the q-state Potts-glass defined by the Hamiltonian

$$\mathcal{H} = -\sum_{(i,j)} J_{i,j}\delta_{s_i s_j} \,. \tag{1.1}$$

The sum runs over all nearest neighbour pairs of spins on the lattice. The spins can be in q different states. The spin–spin interaction energies $J_{i,j}$ are quenched random variables. Their distribution is chosen to be bimodal:

$$P(J_{i,j}) = \tfrac{1}{2}(\delta(J_{i,j} - J) + \delta(J_{i,j} + J)) \,. \tag{1.2}$$

This form of the distribution of the bonds will be essential for the averaging over the random variables in the glass. It may be of interest to study different distributions and how this affects the critical properties of the model to elucidate if the Gaussian and bimodal Potts-glass belong to different universality classes, which has been suggested by Banavar and Cieplak [7]. In the next section we will show how the obtained series data can be used to compute the high temperature series expansion (HTE) for a model with this kind of distribution, as well as others. In this paper we present a method which enables us to compute thermodynamic functions for an arbitrary number of states q of the model. This opens the possibility of studying critical behaviour of Potts-models in dependency on the number of possible orientations of the spins.

0305-4470/94/041079+19$19.50

2. Method

We computed the Helmholtz free energy and the inverse susceptibility.

$$A = [\ln(Z)]_{\mathrm{av}} \tag{2.1}$$

$$\frac{1}{\beta}\chi = \sum_{\langle i,j\rangle}\left[\left\langle\frac{1}{(q-1)}(q\delta_{s_is_j}-1)\right\rangle^2\right]_{\mathrm{av}} . \tag{2.2}$$

The angular brackets refer to thermal averaging, while square brackets refer to an average over the disorder. The Helmholtz free energy and the *inverse* susceptibility have a star-graph expansion

$$\Psi(\mathcal{L}) = \sum_{g\subseteq\mathcal{L}}(g;\mathcal{L})\Psi(g) . \tag{2.3}$$

The sum runs over all star-graphs g embeddable in the lattice $\mathcal{L}$. Computing the sum for all graphs up to N bonds yields the result exact up to this order in the expansion variable. The multiplicity of a graph on the lattice is the combinatorial factor $(g;\mathcal{L})$ which is known as the weak-embedding constant of the graph g in the lattice $\mathcal{L}$. $\Psi(g)$ is the weight function of the graph which is a multinomial in the expansion variables $v_{i,j}$ and in q. The star-graphs which contribute are the same as for the Potts-ferromagnet. For a list of them up to O(10) see our companion paper about the Potts-ferromagnet [8]. The partition function can be written as

$$Z = \mathrm{Tr}\prod_{\langle i,j\rangle}\frac{\mathrm{e}^{K_{i,j}}+(q-1)}{q}\left(1+v_{i,j}(q\delta_{s_is_j}-1)\right) \tag{2.4}$$

where the expansion variables are

$$v_{i,j} := \frac{\mathrm{e}^{K_{i,j}}-1}{\mathrm{e}^{K_{i,j}}+(q-1)} \qquad K_{i,j} := \beta J_{i,j} . \tag{2.5}$$

Notice that due to the disorder the $v_{i,j}$ have to be individual variables until the averaging over the bond-distribution. This means we have to deal with a multinomial expansion. The susceptibility is computed as follows. The expectation value in (2.2) can be seen as the sum of matrix elements $M_{i,j}$,

$$\frac{1}{\beta}\chi =: \sum_{i,j} M_{i,j} = \left[\frac{C_{i,j}}{Z}\right]_{\mathrm{av}} \tag{2.6}$$

where the correlation functions $C_{i,j}$ are

$$C_{i,j} = \mathrm{Tr}\prod_{\langle k,l\rangle}\frac{\mathrm{e}^{K}+(q-1)}{q}\left(1+v_{k,l}(q\delta_{s_ks_l}-1)\right)\frac{1}{q-1}(q\delta_{s_is_j}-1) . \tag{2.7}$$

The $C_{i,j}$ and the partition function Z can be computed using the method described in the companion paper [8]. By this means we could compute these functions for arbitrary number of states q. In contrast to the pure Potts-model one has to use a multinomial expansion in the $v_{i,j}$. After computing the multinomial expansion to the desired order one has to perform the averaging, which is one of the time-consuming parts of the computation. The averaging process is similar to a kind of variable transformation. The individual variables $v_{i,j}^n$ are transformed into the averaged variable N. Different variables $v_{i,j}$ and $v_{k,l}$ are independent random variables so the averaging process for a function Q is

$$\begin{aligned}\left[Q(v_{i,j}, v_{k,l}, \ldots)\right]_{\mathrm{av}} &= \int \mathrm{d} \prod_{\langle\alpha,\beta\rangle} J_{\alpha,\beta} P(J_{\alpha,\beta}) Q(v_{i,j}, v_{k,l}, \ldots) \\ &= \prod_{\langle\alpha,\beta\rangle} \int \mathrm{d}J_{\alpha,\beta}\, P(J_{\alpha,\beta}) Q(v_{i,j}, v_{k,l}, \ldots)\,. \end{aligned} \tag{2.8}$$

If the general function Q is a product of powers of the $v_{i,j}$ this can be further factorized. Finally these 'elementary' integrals can be seen as new 'variables'.

$$\left[v_{i,j}^n\right]_{\mathrm{av}} = \int \mathrm{d}J_{i,j} P(J_{i,j}) v_{i,j}^n =: N\,. \tag{2.9}$$

The matrix inversion is a subtle point where one must not confuse powers of averaged variables and averaged powers of variables. During the matrix inversion multiplications of averaged variables have to be performed, one has to remember that the averaging has to be done *before* the matrix inversion. In the Ising case this is all the same, however for the general q-state Potts-glass we have

$$\left[v_{i,j}^a\right]_{\mathrm{av}}^b \neq \left[v_{i,j}^b\right]_{\mathrm{av}}^a\,. \tag{2.10}$$

There is an equality in the case of the Ising-glass but we have an inequality for a general q-state Potts-glass. The reason for this is the asymmetry of the family of functions, which define the expansion variables (2.5). In the Ising case v reduces to the *symmetric* hyperbolic tangent. The inequality (2.10) in these new variables reads as

$$A^b \neq B^a\,.$$

The matrix elements are now functions of these new 'variables'. These variables are themselves functions of q and K^2. Notice that the odd terms in K vanish after the averaging. This is an effect of the bimodal distribution. In contrast to the Ising-spin glass where only even powers of the expansion variable $v = \tanh$ contribute, due to the symmetry of the tanh, here all powers of v do contribute. All in all it is possible to compute the elementary integrals in advance to have them at hand for performing the averaging. The inverse susceptibility is computed by inverting the matrix M after a LU-decomposition. The weight function of a graph g is defined as

$$\Psi(g) = \sum_{i,j} (M^{-1})_{i,j} - \dim(M) - \sum_{g_s \subseteq g} \Psi(g_s) \tag{2.11}$$

where the sum to be subtracted runs over all *star*-subgraphs of g. If we want to use different distributions this has to be taken into account at this point of the computation. Averaging over a Gaussian distribution is not so simple because one can not *analytically* solve the integrals (2.8). However it is straightforward to solve the integrals for any non-symmetric distribution

$$P(J_{i,j}) = p\delta(J_{i,j} - J) + (1-p)\delta(J_{i,j} + J)$$

which will result in non-vanishing odd exponents in K in the series. This opens up the possibility of studying *asymmetric* distributions of the random bonds and, more generally, the influence of *different distributions* in the model.

3. Results

We present here the free energy of the q-state Potts-glass on simple cubic lattices as a function of the number of states q and the dimension d. Notice that the weight functions obtained are polynomials in K^2 and q. The dimension, d, of the simple cubic lattice enters only in the lattice constants. Our results agree with the known result in $(q; d) = (2; 3)$ [9]. The series

$$A(q, d, K) = 1 + \sum_i a_i(q, d) K^{2i} \tag{3.1}$$

for the special cases $q = 2$–6 and $d = 2$–10 are presented in appendix A. The susceptibilities of the q-state Potts-glass

$$\frac{1}{\beta}\chi(q, d, K) = 1 + \sum_i s_i(q, d) K^{2i} \tag{3.2}$$

for the special cases $q = 2$–6 and $d = 2$–10 are presented in appendix B. The Ising case $(q; d) = (2; 2, 3, 4)$ agrees with the result of Singh and Chakravarty [9], which we could verify using the star-graph expansion of the Ising-spin glass in the expansion variable w. For comparison one has to transform the Ising series in $w := \tanh^2 K$ into a series in K^2. The two-state Potts series in K^2 has to be scaled by a factor of $\frac{1}{2}$ for each bond, because

$$\frac{e^K - 1}{e^K + 1} = \tanh K/2\,. \tag{3.3}$$

This was a proof that we did not miss any graph and that the embedding constants were correct. For a further check of our result in the special case $(q = 3; d = 3)$ we computed the susceptibility directly by a simple expansion in connected graphs up to O(8) on the square lattice. This avoided the matrix-inversion process. However, we had to compute the weight functions of eight graphs instead of two graphs as in the star-graph expansion. The resulting series is a function of K. We found the same result as with the star-graph method. The case $(d; q) = (3; 3)$ is especially interesting. There are Monte Carlo (MC) data available for comparison of the result. In a previous paper by one of the authors a comparison of the MC data with some earlier results by Singh [4] did show some deviations of the obtained results for temperatures below 0.7. This could not be understood by finite-size effects in the MC data. The conclusion was that the series data showed a systematic deviation which was not understood. Our own results are already different from the results of Singh in the fourth order of the series. Our own data show excellent agreement with the MC data. A comparison of Monte Carlo data from [3] and the series itself in ninth and tenth order as well as some Padé approximants are shown in figure 1. Comparing the series in ninth and tenth order shows that the series has converged for temperatures above 0.6.

4. Outlook

The analysis of the glass series is extremely difficult due to the changing sign of the coefficients. Ratio analysis methods completely fail due to irregular changes in the sign of the coefficients. Extending the series to higher order is a first step to obtain improved data. We think that it is possible, without too much labour, to extend the series by two orders. It is planned to do so. The complete analysis of the series for the three-state Potts-glass in different dimensions will be published elsewhere. The detailed analysis of the series for different values of q and d will be presented in a subsequent paper.

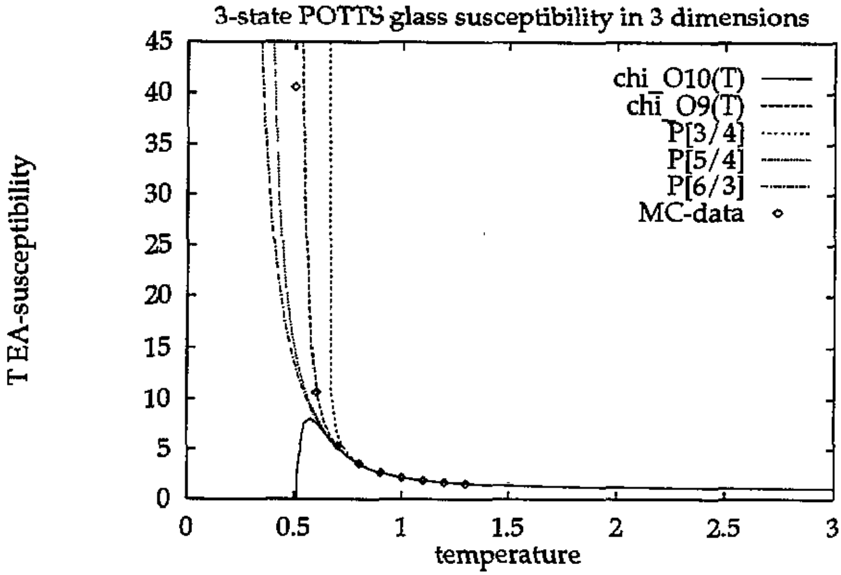

Figure 1. Monte Carlo data from [3] in comparison to the series and some Padé approximants for $(d, q) = (3, 3)$.

Acknowledgment

This work was supported by the Deutsche Forschungsgemeinschaft under grant No SFB 262/D1.

Appendix A

$$A(q=2,d=2) = 1 - \frac{K^8}{512} + \frac{K^{10}}{768} - \frac{49}{184\,320}K^{12} + \frac{59}{387\,072}K^{14} - \frac{340\,199}{1\,238\,630\,400}K^{16} + \frac{150\,631}{437\,944\,320}K^{18} - \frac{15\,660\,670\,639}{47\,823\,519\,744\,000}K^{20} + \mathrm{O}\left(K^{22}\right)$$

$$A(q=2,d=3) = 1 - \frac{3}{512}K^8 + \frac{K^{10}}{256} + \frac{3611}{61\,440}K^{12} - \frac{3719}{64\,512}K^{14} + \frac{13\,083\,421}{412\,876\,800}K^{16} - \frac{10\,804\,331}{1\,021\,870\,080}K^{18} - \frac{9\,599\,950\,667}{7\,970\,586\,624\,000}K^{20} + \mathrm{O}\left(K^{22}\right)$$

$$A(q=2,d=4) = 1 - \frac{3}{256}K^8 + \frac{K^{10}}{128} + \frac{15\,911}{30\,720}K^{12} - \frac{33\,079}{64\,512}K^{14} + \frac{59\,718\,121}{206\,438\,400}K^{16} - \frac{1\,573\,885}{14\,598\,144}K^{18} + \frac{5\,348\,947\,991}{7\,970\,586\,624\,000}K^{20} + \mathrm{O}\left(K^{22}\right)$$

$$A(q=2,d=5) = 1 - \frac{5}{256}K^8 + \frac{5}{384}K^{10} + \frac{36\,851}{18\,432}K^{12} - \frac{6005}{3024}K^{14} + \frac{139\,563\,901}{123\,863\,040}K^{16} - \frac{131\,790\,515}{306\,561\,024}K^{18} - \frac{6\,324\,724\,733}{597\,793\,996\,800}K^{20} + \mathrm{O}\left(K^{22}\right)$$

$$A(q=2,d=6)=1-\frac{15}{512}K^8+\frac{5}{256}K^{10}+\frac{66\,431}{12\,288}K^{12}-\frac{693\,965}{129\,024}K^{14}$$
$$+\frac{252\,620\,761}{82\,575\,360}K^{16}-\frac{240\,917\,951}{204\,374\,016}K^{18}-\frac{295\,623\,814\,099}{3\,188\,234\,649\,600}K^{20}+O\left(K^{22}\right)$$

$$A(q=2,d=7)=1-\frac{21}{512}K^8+\frac{7}{256}K^{10}+\frac{732\,557}{61\,440}K^{12}-\frac{109\,433}{9216}K^{14}$$
$$+\frac{398\,888\,701}{58\,982\,400}K^{16}-\frac{382\,468\,283}{145\,981\,440}K^{18}-\frac{419\,156\,387\,957}{1\,138\,655\,232\,000}K^{20}+O\left(K^{22}\right)$$

$$A(q=2,d=8)=1-\frac{7}{128}K^8+\frac{7}{192}K^{10}+\frac{1\,060\,577}{46\,080}K^{12}-\frac{317\,083}{13\,824}K^{14}$$
$$+\frac{578\,367\,721}{44\,236\,800}K^{16}-\frac{556\,441\,511}{109\,486\,080}K^{18}-\frac{1\,787\,248\,358\,509}{1\,707\,982\,848\,000}K^{20}+O\left(K^{22}\right)$$

$$A(q=2,d=9)=1-\frac{9}{128}K^8+\frac{3}{64}K^{10}+\frac{207\,011}{5120}K^{12}-\frac{108\,361}{2688}K^{14}$$
$$+\frac{791\,057\,821}{34\,406\,400}K^{16}-\frac{21\,795\,361}{2\,433\,024}K^{18}-\frac{812\,753\,559\,271}{332\,107\,776\,000}K^{20}+O\left(K^{22}\right)$$

$$A(q=2,d=10)=1-\frac{45}{512}K^8+\frac{15}{256}K^{10}+\frac{271\,151}{4096}K^{12}-\frac{2\,839\,745}{43\,008}K^{14}$$
$$+\frac{1\,036\,959\,001}{27\,525\,120}K^{16}-\frac{1\,001\,656\,655}{68\,124\,672}K^{18}-\frac{5\,338\,194\,086\,839}{1\,062\,744\,883\,200}K^{20}+O\left(K^{22}\right)$$

$$A(q=3,d=2)=1-\frac{5}{17\,496}K^8+\frac{43}{472\,392}K^{10}-\frac{53}{34\,012\,224}K^{12}-\frac{157}{133\,923\,132}K^{14}$$
$$-\frac{624\,931}{1\,028\,529\,653\,760}K^{16}+\frac{52\,357\,057}{305\,473\,307\,166\,720}K^{18}$$
$$+\frac{76\,274\,043\,983}{476\,538\,359\,180\,083\,200}K^{20}+O\left(K^{22}\right)$$

$$A(q=3,d=3)=1-\frac{5}{5\,832}K^8+\frac{43}{157\,464}K^{10}+\frac{20\,443}{11\,337\,408}K^{12}-\frac{51\,881}{59\,521\,392}K^{14}$$
$$+\frac{176\,006\,519}{1\,028\,529\,653\,760}K^{16}+\frac{696\,758\,011}{305\,473\,307\,166\,720}K^{18}$$
$$-\frac{1\,139\,377\,063\,943}{95\,307\,671\,836\,016\,640}K^{20}+O\left(K^{22}\right)$$

$$A(q=3,d=4)=1-\frac{5}{2\,916}K^8+\frac{43}{78\,732}K^{10}+\frac{89\,323}{5\,668\,704}K^{12}-\frac{343\,825}{44\,641\,044}K^{14}$$
$$+\frac{795\,629\,719}{514\,264\,826\,880}K^{16}+\frac{1\,733\,601\,203}{152\,736\,653\,583\,360}K^{18}$$
$$-\frac{5\,664\,919\,221\,079}{47\,653\,835\,918\,008\,320}K^{20}+O\left(K^{22}\right)$$

$$A(q=3,d=5)=1-\frac{25}{8748}K^8+\frac{215}{236\,196}K^{10}+\frac{1\,032\,935}{17\,006\,112}K^{12}-\frac{7\,983\,245}{267\,846\,264}K^{14}$$
$$+\frac{618\,998\,269}{102\,852\,965\,376}K^{16}+\frac{1\,089\,200\,249}{30\,547\,330\,716\,672}K^{18}$$
$$-\frac{23\,851\,700\,186\,177}{47\,653\,835\,918\,008\,320}K^{20}+O\left(K^{22}\right)$$

$$A(q=3,d=6)=1-\frac{25}{5832}K^8+\frac{215}{157\,464}K^{10}+\frac{1\,861\,175}{11\,337\,408}K^{12}-\frac{300\,275}{3\,720\,087}K^{14}$$

$$+\frac{3\,360\,101\,783}{205\,705\,930\,752}K^{16}+\frac{5\,298\,756\,643}{61\,094\,661\,433\,344}K^{18}$$
$$-\frac{139\,289\,189\,012\,563}{95\,307\,671\,836\,016\,640}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=3,d=7)=1-\frac{35}{5832}K^{8}+\frac{301}{157\,464}K^{10}+\frac{4\,103\,869}{11\,337\,408}K^{12}-\frac{4\,545\,623}{25\,509\,168}K^{14}$$
$$+\frac{5\,304\,950\,647}{146\,932\,807\,680}K^{16}+\frac{7\,827\,068\,891}{43\,639\,043\,880\,960}K^{18}$$
$$-\frac{235\,427\,651\,800\,931}{68\,076\,908\,454\,297\,600}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=3,d=8)=1-\frac{35}{4374}K^{8}+\frac{301}{118\,098}K^{10}+\frac{5\,940\,781}{8\,503\,056}K^{12}-\frac{3\,292\,799}{9\,565\,938}K^{14}$$
$$+\frac{854\,615\,711}{12\,244\,400\,640}K^{16}+\frac{401\,945\,833}{1\,212\,195\,663\,360}K^{18}$$
$$-\frac{1\,617\,205\,685\,587}{226\,923\,028\,180\,992}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=3,d=9)=1-\frac{5}{486}K^{8}+\frac{43}{13\,122}K^{10}+\frac{1\,159\,483}{944\,784}K^{12}-\frac{1\,000\,285}{1\,653\,372}K^{14}$$
$$+\frac{10\,519\,874\,039}{85\,710\,804\,480}K^{16}+\frac{14\,375\,162\,443}{25\,456\,108\,930\,560}K^{18}$$
$$-\frac{105\,704\,172\,342\,887}{7\,942\,305\,986\,334\,720}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=3,d=10)=1-\frac{25}{1944}K^{8}+\frac{215}{52\,488}K^{10}+\frac{7\,593\,335}{3\,779\,136}K^{12}-\frac{14\,745\,655}{14\,880\,348}K^{14}$$
$$+\frac{13\,789\,948\,567}{68\,568\,643\,584}K^{16}+\frac{18\,394\,943\,747}{20\,364\,887\,144\,448}K^{18}$$
$$-\frac{733\,277\,189\,506\,451}{31\,769\,223\,945\,338\,880}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=4,d=2)=1-\frac{3}{131\,072}K^{8}-\frac{13}{524\,288}K^{10}+\frac{599}{62\,914\,560}K^{12}$$
$$-\frac{9151}{8\,455\,716\,864}K^{14}+\frac{7\,530\,541}{27\,058\,293\,964\,800}K^{16}$$
$$-\frac{21\,203\,773}{214\,301\,688\,201\,216}K^{18}+\frac{113\,581\,424\,923}{8\,357\,765\,839\,847\,424\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=4,d=3)=1-\frac{9}{131\,072}K^{8}-\frac{39}{524\,288}K^{10}+\frac{757}{10\,485\,760}K^{12}$$
$$+\frac{185\,771}{2\,818\,572\,288}K^{14}-\frac{265\,847\,039}{9\,019\,431\,321\,600}K^{16}$$
$$+\frac{1\,373\,010\,313}{357\,169\,480\,335\,360}K^{18}+\frac{2\,698\,038\,712\,301}{5\,571\,843\,893\,231\,616\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=4,d=4)=1-\frac{9}{65\,536}K^{8}-\frac{39}{262\,144}K^{10}+\frac{4589}{10\,485\,760}K^{12}+\frac{852\,437}{1\,409\,286\,144}K^{14}$$
$$-\frac{1\,220\,887\,139}{4\,509\,715\,660\,800}K^{16}+\frac{6\,707\,042\,467}{178\,584\,740\,167\,680}K^{18}$$
$$+\frac{1\,930\,825\,032\,119}{696\,480\,486\,653\,952\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=4,d=5)=1-\frac{15}{65\,536}K^8-\frac{65}{262\,144}K^{10}+\frac{307}{196\,608}K^{12}+\frac{9\,954\,235}{4\,227\,858\,432}K^{14}$$
$$-\frac{2\,857\,589\,759}{2\,705\,829\,396\,480}K^{16}+\frac{15\,896\,077\,597}{107\,150\,844\,100\,608}K^{18}$$
$$+\frac{15\,194\,363\,423\,171}{1\,671\,553\,167\,969\,484\,800}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=4,d=6)=1-\frac{45}{131\,072}K^8-\frac{195}{524\,288}K^{10}+\frac{17\,219}{4\,194\,304}K^{12}+\frac{18\,005\,005}{2\,818\,572\,288}K^{14}$$
$$-\frac{5\,175\,954\,899}{1\,803\,886\,264\,320}K^{16}+\frac{28\,940\,115\,703}{71\,433\,896\,067\,072}K^{18}$$
$$+\frac{12\,501\,322\,460\,593}{557\,184\,389\,323\,161\,600}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=4,d=7)=1-\frac{63}{131\,072}K^8-\frac{273}{524\,288}K^{10}+\frac{93\,709}{10\,485\,760}K^{12}+\frac{5\,682\,899}{402\,653\,184}K^{14}$$
$$-\frac{8\,175\,982\,559}{1\,288\,490\,188\,800}K^{16}+\frac{9\,167\,831\,357}{10\,204\,842\,295\,296}K^{18}$$
$$+\frac{37\,039\,560\,947\,321}{795\,977\,699\,033\,088\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=4,d=8)=1-\frac{21}{32\,768}K^8-\frac{91}{131\,072}K^{10}+\frac{269\,423}{15\,728\,640}K^{12}+\frac{8\,236\,541}{301\,989\,888}K^{14}$$
$$-\frac{11\,857\,672\,739}{966\,367\,641\,600}K^{16}+\frac{66\,593\,200\,843}{38\,268\,158\,607\,360}K^{18}$$
$$+\frac{6\,399\,565\,978\,297}{74\,622\,909\,284\,352\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=4,d=9)=1-\frac{27}{32\,768}K^8-\frac{117}{131\,072}K^{10}+\frac{39\,273}{1\,310\,720}K^{12}+\frac{11\,261\,927}{234\,881\,024}K^{14}$$
$$-\frac{16\,221\,025\,439}{751\,619\,276\,800}K^{16}+\frac{91\,202\,247\,877}{29\,764\,123\,361\,280}K^{18}$$
$$+\frac{67\,364\,961\,883\,151}{464\,320\,324\,435\,968\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=4,d=10)=1-\frac{135}{131\,072}K^8-\frac{585}{524\,288}K^{10}+\frac{205\,197}{4\,194\,304}K^{12}+\frac{73\,795\,285}{939\,524\,096}K^{14}$$
$$-\frac{21\,266\,040\,659}{601\,295\,421\,440}K^{16}+\frac{119\,666\,297\,887}{23\,811\,298\,689\,024}K^{18}$$
$$+\frac{42\,718\,139\,721\,223}{185\,728\,129\,774\,387\,200}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=5,d=2)=1+\frac{49}{1\,562\,500}K^8-\frac{571}{23\,437\,500}K^{10}+\frac{188\,723}{70\,312\,500\,000}K^{12}$$
$$+\frac{233\,701}{659\,179\,687\,500}K^{14}-\frac{1\,671\,031\,127}{29\,531\,250\,000\,000\,000}K^{16}$$
$$+\frac{1\,577\,304\,771\,247}{73\,089\,843\,750\,000\,000\,000}K^{18}-\frac{519\,371\,629\,147\,109}{81\,442\,968\,750\,000\,000\,000\,000}K^{20}$$
$$+\mathrm{O}\left(K^{22}\right)$$

$$A(q=5,d=3)=1+\frac{147}{1\,562\,500}K^8-\frac{571}{7\,812\,500}K^{10}-\frac{314\,201}{4\,687\,500\,000}K^{12}$$

$$+\frac{22\,820\,387}{439\,453\,125\,000}K^{14}-\frac{89\,854\,536\,887}{9\,843\,750\,000\,000\,000}K^{16}$$
$$+\frac{10\,412\,642\,248\,807}{24\,363\,281\,250\,000\,000\,000}K^{18}+\frac{248\,117\,811\,465\,643}{27\,147\,656\,250\,000\,000\,000\,000}K^{20}$$
$$+\mathrm{O}\left(K^{22}\right)$$

$$A(q=5,d=4)=1+\frac{147}{781\,250}K^{8}-\frac{571}{3\,906\,250}K^{10}-\frac{1\,496\,969}{2\,343\,750\,000}K^{12}$$
$$+\frac{24\,729\,623}{54\,931\,640\,625}K^{14}-\frac{386\,094\,625\,367}{4\,921\,875\,000\,000\,000}K^{16}$$
$$+\frac{34\,522\,864\,866\,079}{12\,181\,640\,625\,000\,000\,000}K^{18}+\frac{2\,427\,736\,657\,430\,203}{13\,573\,828\,125\,000\,000\,000\,000}K^{20}$$
$$+\mathrm{O}\left(K^{22}\right)$$

$$A(q=5,d=5)=1+\frac{49}{156\,250}K^{8}-\frac{571}{2\,343\,750}K^{10}-\frac{17\,552\,797}{7\,031\,250\,000}K^{12}$$
$$+\frac{228\,761\,717}{131\,835\,937\,500}K^{14}-\frac{890\,391\,296\,567}{2\,953\,125\,000\,000\,000}K^{16}$$
$$+\frac{73\,907\,972\,623\,063}{7\,308\,984\,375\,000\,000\,000}K^{18}+\frac{34\,624\,512\,503\,559\,287}{40\,721\,484\,375\,000\,000\,000\,000}K^{20}$$
$$+\mathrm{O}\left(K^{22}\right)$$

$$A(q=5,d=6)=1+\frac{147}{312\,500}K^{8}-\frac{571}{1\,562\,500}K^{10}-\frac{31\,774\,861}{4\,687\,500\,000}K^{12}$$
$$+\frac{206\,175\,031}{43\,945\,312\,500}K^{14}-\frac{1\,602\,744\,550\,487}{1\,968\,750\,000\,000\,000}K^{16}$$
$$+\frac{128\,567\,965\,519\,759}{4\,872\,656\,250\,000\,000\,000}K^{18}+\frac{73\,225\,164\,666\,379\,463}{27\,147\,656\,250\,000\,000\,000\,000}K^{20}$$
$$+\mathrm{O}\left(K^{22}\right)$$

$$A(q=5,d=7)=1+\frac{1029}{1\,562\,500}K^{8}-\frac{3997}{7\,812\,500}K^{10}-\frac{351\,057\,259}{23\,437\,500\,000}K^{12}$$
$$+\frac{4\,547\,784\,689}{439\,453\,125\,000}K^{14}-\frac{2\,523\,154\,387\,127}{1\,406\,250\,000\,000\,000}K^{16}$$
$$+\frac{198\,502\,843\,556\,167}{3\,480\,468\,750\,000\,000\,000}K^{18}+\frac{37\,090\,963\,765\,922\,633}{5\,429\,531\,250\,000\,000\,000\,000}K^{20}$$
$$+\mathrm{O}\left(K^{22}\right)$$

$$A(q=5,d=8)=1+\frac{343}{390\,625}K^{8}-\frac{3997}{5\,859\,375}K^{10}-\frac{20\,350\,771}{703\,125\,000}K^{12}$$
$$+\frac{3\,292\,667\,392}{164\,794\,921\,875}K^{14}-\frac{3\,651\,620\,806\,487}{1\,054\,687\,500\,000\,000}K^{16}$$
$$+\frac{283\,712\,606\,732\,287}{2\,610\,351\,562\,500\,000\,000}K^{18}+\frac{303\,631\,005\,538\,785\,757}{20\,360\,742\,187\,500\,000\,000\,000}K^{20}$$
$$+\mathrm{O}\left(K^{22}\right)$$

$$A(q=5,d=9)=1+\frac{441}{390\,625}K^{8}-\frac{1713}{1\,953\,125}K^{10}-\frac{3\,974\,629}{78\,125\,000}K^{12}$$
$$+\frac{1\,285\,585\,817}{36\,621\,093\,750}K^{14}-\frac{4\,988\,143\,808\,567}{820\,312\,500\,000\,000}K^{16}$$

$$+\frac{384\,197\,255\,048\,119}{2\,030\,273\,437\,500\,000\,000}K^{18}+\frac{66\,197\,387\,686\,315\,147}{2\,262\,304\,687\,500\,000\,000\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=5,d=10)=1+\frac{441}{312\,500}K^{8}-\frac{1713}{1\,562\,500}K^{10}-\frac{130\,204\,237}{1\,562\,500\,000}K^{12}+\frac{842\,077\,321}{14\,648\,437\,500}K^{14}-\frac{6\,532\,723\,393\,367}{656\,250\,000\,000\,000}K^{16}+\frac{499\,956\,788\,503\,663}{1\,624\,218\,750\,000\,000\,000}K^{18}+\frac{479\,317\,761\,978\,307\,847}{9\,049\,218\,750\,000\,000\,000\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=6,d=2)=1+\frac{5}{124\,416}K^{8}-\frac{235}{15\,116\,544}K^{10}-\frac{173}{2\,176\,782\,336}K^{12}+\frac{26\,305}{68\,568\,643\,584}K^{14}-\frac{4\,324\,427}{131\,651\,795\,681\,280}K^{16}+\frac{8\,484\,779}{977\,514\,582\,933\,504}K^{18}-\frac{1\,926\,864\,601}{15\,249\,227\,493\,762\,662\,400}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=6,d=3)=1+\frac{5}{41\,472}K^{8}-\frac{235}{5\,038\,848}K^{10}-\frac{1859}{26\,873\,856}K^{12}+\frac{35\,075}{1\,269\,789\,696}K^{14}-\frac{325\,546\,661}{131\,651\,795\,681\,280}K^{16}-\frac{90\,504\,451}{977\,514\,582\,933\,504}K^{18}+\frac{184\,378\,767\,581}{7\,624\,613\,746\,881\,331\,200}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=6,d=4)=1+\frac{5}{20\,736}K^{8}-\frac{235}{2\,519\,424}K^{10}-\frac{218\,293}{362\,797\,056}K^{12}+\frac{2\,677\,835}{11\,428\,107\,264}K^{14}-\frac{1\,312\,499\,761}{65\,825\,897\,840\,640}K^{16}-\frac{556\,297\,295}{488\,757\,291\,466\,752}K^{18}+\frac{1\,486\,006\,693\,187}{7\,624\,613\,746\,881\,331\,200}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=6,d=5)=1+\frac{25}{62\,208}K^{8}-\frac{1175}{7\,558\,272}K^{10}-\frac{2\,522\,365}{1\,088\,391\,168}K^{12}+\frac{120\,425}{133\,923\,132}K^{14}-\frac{991\,277\,527}{13\,165\,179\,568\,128}K^{16}-\frac{2\,286\,540\,325}{488\,757\,291\,466\,752}K^{18}+\frac{25\,349\,092\,213}{31\,769\,223\,945\,338\,880}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=6,d=6)=1+\frac{25}{41\,472}K^{8}-\frac{1175}{5\,038\,848}K^{10}-\frac{1\,514\,555}{241\,864\,704}K^{12}+\frac{18\,491\,875}{7\,618\,738\,176}K^{14}-\frac{5\,309\,545\,121}{26\,330\,359\,136\,256}K^{16}-\frac{1\,812\,417\,965}{139\,644\,940\,419\,072}K^{18}+\frac{7\,165\,847\,690\,017}{3\,049\,845\,498\,752\,532\,480}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=6,d=7)=1+\frac{35}{41\,472}K^{8}-\frac{1645}{5\,038\,848}K^{10}-\frac{10\,017\,511}{725\,594\,112}K^{12}+\frac{8\,732\,965}{1\,632\,586\,752}K^{14}-\frac{8\,319\,637\,381}{18\,807\,399\,383\,040}K^{16}-\frac{4\,052\,680\,163}{139\,644\,940\,419\,072}K^{18}+\frac{6\,168\,522\,162\,811}{1\,089\,230\,535\,268\,761\,600}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$A(q=6,d=8)=1+\frac{35}{31\,104}K^8-\frac{1645}{3\,779\,136}K^{10}-\frac{14\,500\,451}{544\,195\,584}K^{12}+\frac{25\,278\,175}{2\,448\,880\,128}K^{14}$$
$$-\frac{49\,399\,627}{58\,047\,528\,960}K^{16}-\frac{657\,534\,359}{11\,637\,078\,368\,256}K^{18}$$
$$+\frac{6\,489\,387\,194\,629}{544\,615\,267\,634\,380\,800}K^{20}+O\left(K^{22}\right)$$

$$A(q=6,d=9)=1+\frac{5}{3\,456}K^8-\frac{235}{419\,904}K^{10}-\frac{943\,331}{20\,155\,392}K^{12}+\frac{2\,877\,655}{158\,723\,712}K^{14}$$
$$-\frac{16\,362\,961\,061}{10\,970\,982\,973\,440}K^{16}-\frac{8\,132\,772\,355}{81\,459\,548\,577\,792}K^{18}$$
$$+\frac{7\,215\,875\,465\,803}{317\,692\,239\,453\,388\,800}K^{20}+O\left(K^{22}\right)$$

$$A(q=6,d=10)=1+\frac{25}{13\,824}K^8-\frac{1175}{1\,679\,616}K^{10}-\frac{18\,532\,865}{241\,864\,704}K^{12}+\frac{226\,134\,925}{7\,618\,738\,176}K^{14}$$
$$-\frac{21\,396\,192\,481}{8\,776\,786\,378\,752}K^{16}-\frac{53\,487\,847\,675}{325\,838\,194\,311\,168}K^{18}$$
$$+\frac{40\,827\,367\,561\,997}{1\,016\,615\,166\,250\,844\,160}K^{20}+O\left(K^{22}\right).$$

Appendix B

$$\chi(q=2,d=2)=1+K^2+\frac{7}{12}K^4+\frac{121}{360}K^6-\frac{503}{20\,160}K^8+\frac{14\,113}{259\,200}K^{10}$$
$$-\frac{10\,429\,369}{119\,750\,400}K^{12}+\frac{1\,600\,250\,243}{21\,794\,572\,800}K^{14}-\frac{10\,866\,157\,699}{135\,862\,272\,000}K^{16}$$
$$+\frac{270\,124\,340\,104\,831}{3\,201\,186\,852\,864\,000}K^{18}-\frac{277\,787\,631\,213\,179}{3\,455\,826\,716\,160\,000}K^{20}+O\left(K^{22}\right)$$

$$\chi(q=2,d=3)=1+\frac{3}{2}K^2+\frac{13}{8}K^4+\frac{421}{240}K^6+\frac{16\,633}{13\,440}K^8+\frac{1\,471\,441}{1\,209\,600}K^{10}$$
$$+\frac{30\,386\,203}{159\,667\,200}K^{12}+\frac{19\,914\,140\,611}{29\,059\,430\,400}K^{14}-\frac{359\,448\,620\,579}{536\,481\,792\,000}K^{16}$$
$$+\frac{2\,040\,961\,668\,733\,681}{2\,134\,124\,568\,576\,000}K^{18}-\frac{1\,138\,371\,578\,707\,127\,057}{810\,967\,336\,058\,880\,000}K^{20}$$
$$+O\left(K^{22}\right)$$

$$\chi(q=2,d=4)=1+2K^2+\frac{19}{6}K^4+\frac{901}{180}K^6+\frac{66\,529}{10\,080}K^8+\frac{8\,653\,951}{907\,200}K^{10}$$
$$+\frac{79\,665\,529}{7\,484\,400}K^{12}+\frac{42\,045\,957\,647}{2\,724\,321\,600}K^{14}+\frac{70\,896\,133\,010\,449}{5\,230\,697\,472\,000}K^{16}$$
$$+\frac{37\,251\,530\,683\,690\,951}{1\,600\,593\,426\,432\,000}K^{18}+\frac{3\,530\,761\,140\,143\,918\,867}{304\,112\,751\,022\,080\,000}K^{20}$$
$$+O\left(K^{22}\right)$$

$$\chi(q=2,d=5)=1+\frac{5}{2}K^2+\frac{125}{24}K^4+\frac{1561}{144}K^6+\frac{164\,305}{8064}K^8+\frac{28\,903\,921}{725\,760}K^{10}$$
$$+\frac{193\,679\,591}{2\,737\,152}K^{12}+\frac{2\,355\,759\,658\,981}{17\,435\,658\,240}K^{14}$$

$$+\frac{192\,521\,240\,330\,333}{836\,911\,595\,520}K^{16}+\frac{80\,107\,290\,281\,739\,463}{182\,924\,963\,020\,800}K^{18}$$
$$+\frac{69\,465\,684\,178\,239\,879\,449}{97\,316\,080\,327\,065\,600}K^{20}+O\left(K^{22}\right)$$

$$\chi(q=2,d=6)=1+3\,K^2+\frac{31}{4}K^4+\frac{2401}{120}K^6+\frac{325\,081}{6720}K^8+\frac{72\,200\,551}{604\,800}K^{10}$$
$$+\frac{1\,603\,837\,679}{5\,702\,400}K^{12}+\frac{4\,952\,967\,477\,413}{7\,264\,857\,600}K^{14}$$
$$+\frac{5\,525\,139\,364\,576\,681}{3\,487\,131\,648\,000}K^{16}+\frac{580\,338\,903\,481\,161\,193}{152\,437\,469\,184\,000}K^{18}$$
$$+\frac{888\,309\,236\,286\,633\,925\,339}{101\,370\,917\,007\,360\,000}K^{20}+O\left(K^{22}\right)$$

$$\chi(q=2,d=7)=1+\frac{7}{2}K^2+\frac{259}{24}K^4+\frac{23\,947}{720}K^6+\frac{563\,977}{5760}K^8+\frac{151\,244\,641}{518\,400}K^{10}$$
$$+\frac{57\,952\,603\,567}{68\,428\,800}K^{12}+\frac{31\,037\,436\,849\,511}{12\,454\,041\,600}K^{14}$$
$$+\frac{1\,954\,467\,574\,648\,187}{271\,724\,544\,000}K^{16}+\frac{19\,237\,403\,937\,827\,393\,281}{914\,624\,815\,104\,000}K^{18}$$
$$+\frac{21\,032\,088\,118\,587\,135\,395\,947}{347\,557\,429\,739\,520\,000}K^{20}+O\left(K^{22}\right)$$

$$\chi(q=2,d=8)=1+4\,K^2+\frac{43}{3}K^4+\frac{4621}{90}K^6+\frac{896\,113}{5040}K^8+\frac{281\,458\,591}{453\,600}K^{10}$$
$$+\frac{31\,892\,217\,877}{14\,968\,800}K^{12}+\frac{20\,077\,984\,316\,959}{2\,724\,321\,600}K^{14}$$
$$+\frac{65\,995\,573\,805\,744\,353}{2\,615\,348\,736\,000}K^{16}+\frac{69\,564\,146\,030\,349\,676\,231}{800\,296\,713\,216\,000}K^{18}$$
$$+\frac{45\,195\,784\,802\,948\,030\,955\,449}{152\,056\,375\,511\,040\,000}K^{20}+O\left(K^{22}\right)$$

$$\chi(q=2,d=9)=1+\frac{9}{2}K^2+\frac{147}{8}K^4+\frac{6001}{80}K^6+\frac{1\,336\,609}{4480}K^8$$
$$+\frac{68\,712\,343}{57\,600}K^{10}+\frac{251\,134\,444\,969}{53\,222\,400}K^{12}+\frac{181\,581\,044\,790\,601}{9\,686\,476\,800}K^{14}$$
$$+\frac{3\,512\,413\,815\,271\,921}{47\,443\,968\,000}K^{16}+\frac{208\,641\,422\,625\,802\,317\,601}{711\,374\,856\,192\,000}K^{18}$$
$$+\frac{312\,921\,238\,265\,929\,487\,828\,089}{270\,322\,445\,352\,960\,000}K^{20}+O\left(K^{22}\right)$$

$$\chi(q=2,d=10)=1+5\,K^2+\frac{275}{12}K^4+\frac{7561}{72}K^6+\frac{1\,900\,585}{4032}K^8+\frac{770\,693\,671}{362\,880}K^{10}$$
$$+\frac{45\,491\,782\,159}{4\,790\,016}K^{12}+\frac{16\,875\,249\,571\,573}{396\,264\,960}K^{14}$$
$$+\frac{79\,613\,345\,818\,893\,989}{418\,455\,797\,760}K^{16}+\frac{545\,143\,109\,398\,827\,012\,991}{640\,237\,370\,572\,800}K^{18}$$
$$+\frac{420\,533\,844\,053\,745\,366\,277}{110\,586\,454\,917\,120}K^{20}+O\left(K^{22}\right)$$

$$\chi(q=3,d=2)=1+\frac{4}{9}K^2+\frac{K^4}{9}+\frac{179}{7290}K^6-\frac{21\,997}{3\,674\,160}K^8-\frac{163\,109}{330\,674\,400}K^{10}$$

$$- \frac{112\,499\,111}{130\,947\,062\,400} K^{12} + \frac{80\,769\,775\,817}{214\,491\,288\,211\,200} K^{14} - \frac{8\,987\,675\,215\,037}{154\,433\,727\,512\,064\,000} K^{16} + \frac{1\,930\,856\,220\,172\,331}{47\,256\,720\,618\,691\,584\,000} K^{18} + \frac{52\,195\,111\,586\,882\,107}{161\,617\,984\,515\,925\,217\,280\,000} K^{20} + O\left(K^{22}\right)$$

$$\chi(q=3,d=3) = 1 + \frac{2}{3} K^2 + \frac{17}{54} K^4 + \frac{233}{1620} K^6 + \frac{76\,339}{2\,449\,440} K^8 + \frac{698\,417}{73\,483\,200} K^{10} - \frac{601\,837\,543}{87\,298\,041\,600} K^{12} + \frac{15\,713\,937\,697}{142\,994\,192\,140\,800} K^{14} - \frac{200\,453\,782\,761\,181}{102\,955\,818\,341\,376\,000} K^{16} + \frac{28\,998\,575\,551\,734\,851}{31\,504\,480\,412\,461\,056\,000} K^{18} - \frac{758\,829\,892\,644\,405\,877}{6\,337\,960\,177\,095\,106\,560\,000} K^{20} + O\left(K^{22}\right)$$

$$\chi(q=3,d=4) = 1 + \frac{8}{9} K^2 + \frac{50}{81} K^4 + \frac{19}{45} K^6 + \frac{8959}{40\,824} K^8 + \frac{6\,916\,057}{55\,112\,400} K^{10} + \frac{168\,936\,727}{4\,364\,902\,080} K^{12} + \frac{841\,714\,964\,819}{35\,748\,548\,035\,200} K^{14} - \frac{16\,742\,304\,672\,179}{5\,147\,790\,917\,068\,800} K^{16} + \frac{33\,087\,844\,686\,335\,497}{7\,876\,120\,103\,115\,264\,000} K^{18} - \frac{22\,121\,264\,023\,460\,971\,883}{5\,387\,266\,150\,530\,840\,576\,000} K^{20} + O\left(K^{22}\right)$$

$$\chi(q=3,d=5) = 1 + \frac{10}{9} K^2 + \frac{55}{54} K^4 + \frac{2699}{2916} K^6 + \frac{1\,065\,971}{1\,469\,664} K^8 + \frac{78\,374\,851}{132\,269\,760} K^{10} + \frac{21\,339\,395\,353}{52\,378\,824\,960} K^{12} + \frac{2\,502\,967\,847\,827}{7\,799\,683\,207\,680} K^{14} + \frac{11\,886\,871\,828\,179\,811}{61\,773\,491\,004\,825\,600} K^{16} + \frac{2\,979\,350\,588\,831\,753\,651}{18\,902\,688\,247\,476\,633\,600} K^{18} + \frac{5\,077\,205\,021\,896\,306\,175\,419}{64\,647\,193\,806\,370\,086\,912\,000} K^{20} + O\left(K^{22}\right)$$

$$\chi(q=3,d=6) = 1 + \frac{4}{3} K^2 + \frac{41}{27} K^4 + \frac{1393}{810} K^6 + \frac{2\,172\,307}{1\,224\,720} K^8 + \frac{68\,718\,697}{36\,741\,600} K^{10} + \frac{11\,383\,660\,703}{6\,235\,574\,400} K^{12} + \frac{133\,862\,692\,741\,417}{71\,497\,096\,070\,400} K^{14} + \frac{90\,981\,462\,227\,017\,667}{51\,477\,909\,170\,688\,000} K^{16} + \frac{4\,044\,544\,480\,785\,568\,733}{2\,250\,320\,029\,461\,504\,000} K^{18} + \frac{89\,157\,857\,841\,356\,039\,001\,403}{53\,872\,661\,505\,308\,405\,760\,000} K^{20} + O\left(K^{22}\right)$$

$$\chi(q=3,d=7) = 1 + \frac{14}{9} K^2 + \frac{343}{162} K^4 + \frac{13\,951}{4860} K^6 + \frac{1\,276\,561}{349\,920} K^8 + \frac{147\,958\,097}{31\,492\,800} K^{10} + \frac{72\,352\,024\,243}{12\,471\,148\,800} K^{12} + \frac{149\,734\,717\,015\,339}{20\,427\,741\,734\,400} K^{14} + \frac{18\,783\,150\,185\,466\,583}{2\,101\,139\,149\,824\,000} K^{16} + \frac{4\,582\,116\,589\,017\,978\,307}{409\,149\,096\,265\,728\,000} K^{18} + \frac{16\,066\,330\,190\,788\,372\,543\,693}{1\,184\,014\,538\,578\,206\,720\,000} K^{20} + O\left(K^{22}\right)$$

$$\chi(q=3,d=8)=1+\frac{16}{9}K^2+\frac{76}{27}K^4+\frac{16\,198}{3645}K^6+\frac{6\,145\,619}{918\,540}K^8$$
$$+\frac{839\,913\,451}{82\,668\,600}K^{10}+\frac{70\,332\,954\,271}{4\,676\,680\,800}K^{12}+\frac{1\,208\,339\,575\,646\,297}{53\,622\,822\,052\,800}K^{14}$$
$$+\frac{1\,280\,453\,676\,193\,328\,899}{38\,608\,431\,878\,016\,000}K^{16}+\frac{916\,131\,171\,548\,740\,823}{18\,546\,593\,649\,408\,000}K^{18}$$
$$+\frac{2\,930\,647\,141\,089\,015\,567\,623\,611}{40\,404\,496\,128\,981\,304\,320\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=3,d=9)=1+2K^2+\frac{65}{18}K^4+\frac{1171}{180}K^6+\frac{1\,845\,527}{163\,296}K^8+\frac{161\,251\,019}{8\,164\,800}K^{10}$$
$$+\frac{197\,514\,139\,909}{5\,819\,869\,440}K^{12}+\frac{2\,799\,640\,297\,089\,457}{47\,664\,730\,713\,600}K^{14}$$
$$+\frac{53\,130\,695\,647\,916\,323}{527\,978\,555\,596\,800}K^{16}+\frac{1\,821\,434\,044\,893\,291\,609\,491}{10\,501\,493\,470\,820\,352\,000}K^{18}$$
$$+\frac{426\,482\,236\,042\,861\,733\,245\,939}{1\,436\,604\,306\,808\,224\,153\,600}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=3,d=10)=1+\frac{20}{9}K^2+\frac{365}{81}K^4+\frac{4433}{486}K^6+\frac{4\,394\,417}{244\,944}K^8$$
$$+\frac{260\,388\,739}{7\,348\,320}K^{10}+\frac{67\,103\,829\,659}{969\,978\,240}K^{12}+\frac{646\,647\,478\,248\,833}{4\,766\,473\,071\,360}K^{14}$$
$$+\frac{82\,491\,343\,570\,447\,009}{311\,987\,328\,307\,200}K^{16}+\frac{41\,765\,914\,006\,649\,748\,703}{80\,780\,719\,006\,310\,400}K^{18}$$
$$+\frac{3\,617\,355\,209\,885\,170\,084\,820\,771}{3\,591\,510\,767\,020\,560\,384\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=4,d=2)=1+\frac{K^2}{4}+\frac{5}{96}K^4+\frac{17}{2880}K^6+\frac{89}{64\,512}K^8-\frac{772\,627}{928\,972\,800}K^{10}$$
$$-\frac{2\,968\,307}{98\,099\,527\,680}K^{12}-\frac{6\,191\,639}{174\,356\,582\,400}K^{14}$$
$$+\frac{234\,168\,499\,861}{17\,139\,949\,476\,249\,600}K^{16}-\frac{92\,811\,146\,881\,567}{419\,585\,963\,178\,590\,208\,000}K^{18}$$
$$+\frac{35\,471\,045\,121\,685\,643}{25\,510\,826\,561\,258\,284\,646\,400}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=4,d=3)=1+\frac{3}{8}K^2+\frac{K^4}{8}+\frac{263}{7680}K^6+\frac{9529}{860\,160}K^8+\frac{166\,013}{619\,315\,200}K^{10}$$
$$+\frac{34\,986\,683}{326\,998\,425\,600}K^{12}-\frac{174\,203\,183\,699}{476\,109\,707\,673\,600}K^{14}$$
$$-\frac{8\,293\,308\,762\,371}{114\,266\,329\,841\,664\,000}K^{16}-\frac{19\,390\,257\,975\,283\,387}{279\,723\,975\,452\,393\,472\,000}K^{18}$$
$$+\frac{65\,829\,054\,218\,952\,041}{32\,706\,187\,899\,049\,082\,880\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=4,d=4)=1+\frac{K^2}{2}+\frac{11}{48}K^4+\frac{137}{1440}K^6+\frac{13\,679}{322\,560}K^8+\frac{5\,738\,933}{464\,486\,400}K^{10}$$
$$+\frac{1\,181\,406\,221}{245\,248\,819\,200}K^{12}+\frac{8\,797\,035\,913}{16\,231\,012\,761\,600}K^{14}$$
$$+\frac{1\,512\,258\,309\,097}{21\,424\,936\,845\,312\,000}K^{16}-\frac{67\,789\,674\,953\,749\,747}{209\,792\,981\,589\,295\,104\,000}K^{18}$$

$$-\frac{37\,920\,189\,395\,953\,394\,869}{318\,885\,332\,015\,728\,558\,080\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=4,d=5)=1+\frac{5}{8}K^2+\frac{35}{96}K^4+\frac{923}{4608}K^6+\frac{59\,047}{516\,096}K^8+\frac{2\,893\,619}{53\,084\,160}K^{10}$$
$$+\frac{5\,652\,023\,999}{196\,199\,055\,360}K^{12}+\frac{3\,472\,704\,627\,211}{285\,665\,824\,604\,160}K^{14}$$
$$+\frac{8\,041\,439\,562\,223}{1\,399\,179\,549\,081\,600}K^{16}+\frac{22\,471\,256\,171\,055\,581}{12\,910\,337\,328\,572\,006\,400}K^{18}$$
$$+\frac{169\,609\,616\,396\,689\,037\,909}{255\,108\,265\,612\,582\,846\,464\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=4,d=6)=1+\frac{3}{4}K^2+\frac{17}{32}K^4+\frac{347}{960}K^6+\frac{13\,547}{53\,760}K^8+\frac{49\,385\,213}{309\,657\,600}K^{10}$$
$$+\frac{17\,321\,014\,937}{163\,499\,212\,800}K^{12}+\frac{3\,841\,371\,102\,319}{59\,513\,713\,459\,200}K^{14}$$
$$+\frac{1\,170\,153\,072\,340\,643}{28\,566\,582\,460\,416\,000}K^{16}+\frac{3\,310\,068\,936\,745\,915\,913}{139\,861\,987\,726\,196\,736\,000}K^{18}$$
$$+\frac{180\,145\,149\,523\,654\,963\,171}{12\,505\,307\,137\,871\,708\,160\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=4,d=7)=1+\frac{7}{8}K^2+\frac{35}{48}K^4+\frac{13\,601}{23\,040}K^6+\frac{35\,825}{73\,728}K^8+\frac{100\,159\,373}{265\,420\,800}K^{10}$$
$$+\frac{8\,431\,637\,191}{28\,028\,436\,480}K^{12}+\frac{46\,924\,619\,840\,281}{204\,047\,017\,574\,400}K^{14}$$
$$+\frac{1\,753\,678\,264\,109\,501}{9\,794\,256\,843\,571\,200}K^{16}+\frac{16\,144\,266\,444\,728\,231\,933}{119\,881\,703\,765\,311\,488\,000}K^{18}$$
$$+\frac{3\,757\,027\,163\,201\,721\,572\,833}{36\,444\,037\,944\,654\,692\,352\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=4,d=8)=1+K^2+\frac{23}{24}K^4+\frac{647}{720}K^6+\frac{137\,537}{161\,280}K^8+\frac{180\,969\,413}{232\,243\,200}K^{10}$$
$$+\frac{88\,678\,039\,973}{122\,624\,409\,600}K^{12}+\frac{58\,543\,521\,667\,313}{89\,270\,570\,188\,800}K^{14}$$
$$+\frac{3\,215\,512\,477\,270\,703}{5\,356\,234\,211\,328\,000}K^{16}+\frac{56\,739\,318\,910\,217\,043\,413}{104\,896\,490\,794\,647\,552\,000}K^{18}$$
$$+\frac{78\,287\,468\,601\,547\,831\,295\,723}{159\,442\,666\,007\,864\,279\,040\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=4,d=9)=1+\frac{9}{8}K^2+\frac{39}{32}K^4+\frac{3323}{2560}K^6+\frac{400\,003}{286\,720}K^8+\frac{301\,567\,733}{206\,438\,400}K^{10}$$
$$+\frac{168\,388\,839\,911}{108\,999\,475\,200}K^{12}+\frac{255\,697\,870\,993\,111}{158\,703\,235\,891\,200}K^{14}$$
$$+\frac{4\,949\,425\,470\,082\,591}{2\,929\,905\,893\,376\,000}K^{16}+\frac{163\,788\,166\,981\,913\,598\,353}{93\,241\,325\,150\,797\,824\,000}K^{18}$$
$$+\frac{259\,812\,762\,427\,248\,697\,535\,621}{141\,726\,814\,229\,212\,692\,480\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=4,d=10)=1+\frac{5}{4}K^2+\frac{145}{96}K^4+\frac{1037}{576}K^6+\frac{139\,423}{64\,512}K^8+\frac{473\,067\,533}{185\,794\,560}K^{10}$$
$$+\frac{296\,241\,672\,689}{98\,099\,527\,680}K^{12}+\frac{63\,378\,097\,183\,367}{17\,854\,114\,037\,760}K^{14}$$

$$+\frac{71\,750\,793\,437\,972\,861}{17\,139\,949\,476\,249\,600}K^{16}+\frac{412\,332\,751\,305\,039\,555\,953}{83\,917\,192\,635\,718\,041\,600}K^{18}$$
$$+\frac{737\,056\,505\,277\,379\,805\,874\,899}{127\,554\,132\,806\,291\,423\,232\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=5,d=2)=1+\frac{4}{25}K^2+\frac{67}{1875}K^4+\frac{599}{281\,250}K^6+\frac{581\,591}{393\,750\,000}K^8$$
$$-\frac{44\,561\,621}{177\,187\,500\,000}K^{10}-\frac{31\,869\,466\,757}{584\,718\,750\,000\,000}K^{12}$$
$$-\frac{22\,811\,981\,805\,653}{2\,660\,470\,312\,500\,000\,000}K^{14}-\frac{1\,690\,689\,871\,710\,193}{3\,192\,564\,375\,000\,000\,000\,000}K^{16}$$
$$+\frac{9\,920\,614\,273\,736\,228\,399}{24\,423\,117\,468\,750\,000\,000\,000\,000}K^{18}$$
$$+\frac{1\,230\,710\,248\,335\,633\,957\,239}{9\,280\,784\,638\,125\,000\,000\,000\,000\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=5,d=3)=1+\frac{6}{25}K^2+\frac{91}{1250}K^4+\frac{499}{37\,500}K^6+\frac{1\,644\,023}{262\,500\,000}K^8$$
$$+\frac{5\,407\,621}{16\,875\,000\,000}K^{10}+\frac{48\,918\,820\,603}{389\,812\,500\,000\,000}K^{12}$$
$$-\frac{70\,903\,268\,718\,413}{1\,773\,646\,875\,000\,000\,000}K^{14}-\frac{553\,085\,743\,527\,811}{23\,388\,750\,000\,000\,000\,000}K^{16}$$
$$-\frac{244\,841\,479\,369\,191\,696\,601}{16\,282\,078\,312\,500\,000\,000\,000\,000}K^{18}$$
$$-\frac{151\,867\,855\,779\,658\,468\,119\,533}{30\,935\,948\,793\,750\,000\,000\,000\,000\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=5,d=4)=1+\frac{8}{25}K^2+\frac{46}{375}K^4+\frac{4967}{140\,625}K^6+\frac{3\,440\,279}{196\,875\,000}K^8$$
$$+\frac{337\,040\,731}{88\,593\,750\,000}K^{10}+\frac{468\,822\,360\,571}{292\,359\,375\,000\,000}K^{12}$$
$$+\frac{403\,691\,518\,294\,843}{1\,330\,235\,156\,250\,000\,000}K^{14}+\frac{117\,688\,801\,004\,211\,791}{1\,596\,282\,187\,500\,000\,000\,000}K^{16}$$
$$-\frac{396\,034\,311\,656\,492\,779\,297}{12\,211\,558\,734\,375\,000\,000\,000\,000}K^{18}$$
$$-\frac{465\,613\,345\,521\,210\,872\,978\,093}{23\,201\,961\,595\,312\,500\,000\,000\,000\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=5,d=5)=1+\frac{2}{5}K^2+\frac{139}{750}K^4+\frac{1603}{22\,500}K^6+\frac{1\,232\,779}{31\,500\,000}K^8$$
$$+\frac{972\,605\,779}{70\,875\,000\,000}K^{10}+\frac{45\,521\,312\,201}{6\,682\,500\,000\,000}K^{12}$$
$$+\frac{2\,587\,832\,898\,750\,787}{1\,064\,188\,125\,000\,000\,000}K^{14}+\frac{1\,352\,073\,538\,207\,804\,847}{1\,277\,025\,750\,000\,000\,000\,000}K^{16}$$
$$+\frac{15\,836\,303\,884\,991\,413\,721}{55\,824\,268\,500\,000\,000\,000\,000}K^{18}$$
$$+\frac{1\,940\,645\,383\,825\,401\,239\,205\,139}{18\,561\,569\,276\,250\,000\,000\,000\,000\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=5,d=6)=1+\frac{12}{25}K^2+\frac{163}{625}K^4+\frac{11\,639}{93\,750}K^6+\frac{10\,008\,407}{131\,250\,000}K^8$$

$$+\frac{2\,092\,022\,923}{59\,062\,500\,000}K^{10}+\frac{3\,963\,866\,882\,683}{194\,906\,250\,000\,000}K^{12}+\frac{8\,651\,975\,693\,327\,563}{886\,823\,437\,500\,000\,000}K^{14}+\frac{5\,627\,627\,255\,030\,413\,967}{1\,064\,188\,125\,000\,000\,000\,000}K^{16}+\frac{19\,447\,053\,358\,997\,002\,249\,871}{8\,141\,039\,156\,250\,000\,000\,000\,000}K^{18}+\frac{19\,430\,469\,751\,540\,871\,242\,538\,323}{15\,467\,974\,396\,875\,000\,000\,000\,000\,000}K^{20}+O\left(K^{22}\right)$$

$$\chi(q=5,d=7)=1+\frac{14}{25}K^2+\frac{1309}{3750}K^4+\frac{110\,873}{562\,500}K^6+\frac{15\,167\,351}{112\,500\,000}K^8+\frac{3\,870\,635\,779}{50\,625\,000\,000}K^{10}+\frac{8\,335\,422\,501\,883}{167\,062\,500\,000\,000}K^{12}+\frac{2\,015\,938\,192\,217\,177}{69\,103\,125\,000\,000\,000}K^{14}+\frac{16\,841\,969\,578\,228\,190\,447}{912\,161\,250\,000\,000\,000\,000}K^{16}+\frac{74\,164\,181\,022\,772\,947\,843\,719}{6\,978\,033\,562\,500\,000\,000\,000\,000}K^{18}+\frac{1\,037\,616\,274\,223\,036\,390\,106\,823}{155\,979\,573\,750\,000\,000\,000\,000\,000}K^{20}+O\left(K^{22}\right)$$

$$\chi(q=5,d=8)=1+\frac{16}{25}K^2+\frac{844}{1875}K^4+\frac{8246}{28\,125}K^6+\frac{21\,834\,263}{98\,437\,500}K^8+\frac{6\,511\,657\,147}{44\,296\,875\,000}K^{10}+\frac{2\,244\,642\,470\,509}{20\,882\,812\,500\,000}K^{12}+\frac{48\,651\,306\,369\,678\,427}{665\,117\,578\,125\,000\,000}K^{14}+\frac{41\,925\,950\,220\,168\,707\,279}{798\,141\,093\,750\,000\,000\,000}K^{16}+\frac{2\,828\,667\,648\,780\,389\,149\,627}{79\,295\,835\,937\,500\,000\,000\,000}K^{18}+\frac{295\,675\,561\,714\,394\,655\,325\,716\,307}{11\,600\,980\,797\,656\,250\,000\,000\,000\,000}K^{20}+O\left(K^{22}\right)$$

$$\chi(q=5,d=9)=1+\frac{18}{25}K^2+\frac{141}{250}K^4+\frac{25\,967}{62\,500}K^6+\frac{30\,202\,679}{87\,500\,000}K^8+\frac{10\,246\,169\,011}{39\,375\,000\,000}K^{10}+\frac{27\,386\,310\,324\,091}{129\,937\,500\,000\,000}K^{12}+\frac{96\,073\,452\,779\,183\,203}{591\,215\,625\,000\,000\,000}K^{14}+\frac{92\,320\,751\,077\,874\,410\,031}{709\,458\,750\,000\,000\,000\,000}K^{16}+\frac{545\,923\,791\,770\,341\,383\,311\,543}{5\,427\,359\,437\,500\,000\,000\,000\,000}K^{18}+\frac{63\,727\,725\,851\,586\,804\,499\,728\,799}{793\,229\,456\,250\,000\,000\,000\,000\,000}K^{20}+O\left(K^{22}\right)$$

$$\chi(q=5,d=10)=1+\frac{4}{5}K^2+\frac{259}{375}K^4+\frac{6379}{11\,250}K^6+\frac{8\,093\,227}{15\,750\,000}K^8+\frac{2\,190\,446\,077}{5\,062\,500\,000}K^{10}+\frac{8\,994\,300\,508\,031}{23\,388\,750\,000\,000}K^{12}+\frac{175\,601\,537\,405\,168\,107}{532\,094\,062\,500\,000\,000}K^{14}+\frac{26\,553\,666\,709\,574\,824\,121}{91\,216\,125\,000\,000\,000\,000}K^{16}+\frac{49\,010\,548\,441\,783\,195\,038\,791}{195\,384\,939\,750\,000\,000\,000\,000}K^{18}$$

$$+\frac{2\,050\,636\,537\,132\,959\,114\,128\,857\,939}{9\,280\,784\,638\,125\,000\,000\,000\,000\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=6,d=2)=1+\frac{K^2}{9}+\frac{K^4}{36}+\frac{29}{29\,160}K^6+\frac{15\,173}{14\,696\,640}K^8-\frac{45\,509}{1\,322\,697\,600}K^{10}$$
$$-\frac{8\,466\,133}{261\,894\,124\,800}K^{12}-\frac{2\,437\,028\,729}{428\,982\,576\,422\,400}K^{14}$$
$$-\frac{383\,881\,370\,027}{617\,734\,910\,048\,256\,000}K^{16}-\frac{6\,326\,533\,183\,669}{189\,026\,882\,474\,766\,336\,000}K^{18}$$
$$+\frac{3\,928\,040\,679\,445\,513}{161\,617\,984\,515\,925\,217\,280\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=6,d=3)=1+\frac{K^2}{6}+\frac{11}{216}K^4+\frac{43}{6480}K^6+\frac{37\,237}{9\,797\,760}K^8+\frac{106\,247}{293\,932\,800}K^{10}$$
$$+\frac{29\,374\,181}{349\,192\,166\,400}K^{12}+\frac{151\,369\,847}{51\,997\,888\,051\,200}K^{14}$$
$$-\frac{1\,099\,775\,560\,123}{411\,823\,273\,365\,504\,000}K^{16}-\frac{547\,771\,161\,485\,419}{126\,017\,921\,649\,844\,224\,000}K^{18}$$
$$-\frac{663\,120\,675\,563\,269\,997}{430\,981\,292\,042\,467\,246\,080\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=6,d=4)=1+\frac{2}{9}K^2+\frac{13}{162}K^4+\frac{83}{4860}K^6+\frac{22\,847}{2\,449\,440}K^8+\frac{420\,817}{220\,449\,600}K^{10}$$
$$+\frac{16\,143\,689}{21\,824\,510\,400}K^{12}+\frac{6\,325\,108\,631}{35\,748\,548\,035\,200}K^{14}$$
$$+\frac{464\,001\,918\,059}{7\,919\,678\,333\,952\,000}K^{16}+\frac{3\,791\,274\,297\,347}{2\,864\,043\,673\,860\,096\,000}K^{18}$$
$$-\frac{19\,676\,528\,107\,294\,601}{53\,872\,661\,505\,308\,405\,760\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=6,d=5)=1+\frac{5}{18}K^2+\frac{25}{216}K^4+\frac{389}{11\,664}K^6+\frac{110\,765}{5\,878\,656}K^8$$
$$+\frac{2\,987\,221}{529\,079\,040}K^{10}+\frac{110\,297\,735}{41\,903\,059\,968}K^{12}+\frac{309\,366\,088\,307}{343\,186\,061\,137\,920}K^{14}$$
$$+\frac{19\,699\,480\,809\,017}{49\,418\,792\,803\,860\,480}K^{16}+\frac{8\,316\,645\,408\,551\,141}{75\,610\,752\,989\,906\,534\,400}K^{18}$$
$$+\frac{2\,599\,838\,933\,960\,445\,581}{51\,717\,755\,045\,096\,069\,529\,600}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=6,d=6)=1+\frac{K^2}{3}+\frac{17}{108}K^4+\frac{61}{1080}K^6+\frac{165\,589}{4\,898\,880}K^8+\frac{1\,909\,417}{146\,966\,400}K^{10}$$
$$+\frac{601\,031\,041}{87\,298\,041\,600}K^{12}+\frac{420\,507\,119\,261}{142\,994\,192\,140\,800}K^{14}$$
$$+\frac{313\,903\,331\,396\,069}{205\,911\,636\,682\,752\,000}K^{16}+\frac{38\,495\,368\,603\,103\,051}{63\,008\,960\,824\,922\,112\,000}K^{18}$$
$$+\frac{258\,740\,525\,197\,462\,703}{792\,245\,022\,136\,888\,320\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=6,d=7)=1+\frac{7}{18}K^2+\frac{133}{648}K^4+\frac{7}{80}K^6+\frac{26\,077}{466\,560}K^8+\frac{3\,251\,447}{125\,971\,200}K^{10}$$
$$+\frac{757\,519\,723}{49\,884\,595\,200}K^{12}+\frac{623\,783\,352\,739}{81\,710\,966\,937\,600}K^{14}$$

$$+\frac{261\,917\,378\,553\,031}{58\,831\,896\,195\,072\,000}K^{16}+\frac{39\,564\,701\,894\,219\,927}{18\,002\,560\,235\,692\,032\,000}K^{18}$$
$$+\frac{80\,847\,542\,719\,715\,833\,669}{61\,568\,756\,006\,066\,749\,440\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=6,d=8)=1+\frac{4}{9}K^2+\frac{7}{27}K^4+\frac{929}{7290}K^6+\frac{319\,757}{3\,674\,160}K^8+\frac{2\,195\,413}{47\,239\,200}K^{10}$$
$$+\frac{490\,188\,827}{16\,368\,382\,800}K^{12}+\frac{28\,738\,839\,589}{1\,675\,713\,189\,150}K^{14}$$
$$+\frac{243\,213\,266\,208\,571}{22\,061\,961\,073\,152\,000}K^{16}+\frac{298\,238\,810\,363\,424\,131}{47\,256\,720\,618\,691\,584\,000}K^{18}$$
$$+\frac{30\,383\,363\,726\,591\,093\,089}{7\,346\,272\,023\,451\,146\,240\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=6,d=9)=1+\frac{K^2}{2}+\frac{23}{72}K^4+\frac{383}{2160}K^6+\frac{422\,461}{3\,265\,920}K^8+\frac{2\,544\,989}{32\,659\,200}K^{10}$$
$$+\frac{6\,342\,559\,103}{116\,397\,388\,800}K^{12}+\frac{6\,636\,732\,146\,047}{190\,658\,922\,854\,400}K^{14}$$
$$+\frac{3\,343\,759\,815\,584\,621}{137\,274\,424\,455\,168\,000}K^{16}+\frac{658\,420\,958\,096\,946\,101}{42\,005\,973\,883\,281\,408\,000}K^{18}$$
$$+\frac{123\,118\,293\,227\,101\,338\,673}{11\,050\,802\,360\,063\,262\,720\,000}K^{20}+\mathrm{O}\left(K^{22}\right)$$

$$\chi(q=6,d=10)=1+\frac{5}{9}K^2+\frac{125}{324}K^4+\frac{463}{1944}K^6+\frac{181\,495}{979\,776}K^8+\frac{3\,637\,219}{29\,393\,280}K^{10}$$
$$+\frac{1\,339\,939}{14\,370\,048}K^{12}+\frac{624\,217\,871\,899}{9\,532\,946\,142\,720}K^{14}$$
$$+\frac{135\,775\,887\,908\,329}{2\,745\,488\,489\,103\,360}K^{16}+\frac{147\,422\,788\,814\,458\,099}{4\,200\,597\,388\,328\,140\,800}K^{18}$$
$$+\frac{19\,284\,304\,751\,005\,595\,467}{718\,302\,153\,404\,112\,076\,800}K^{20}+\mathrm{O}\left(K^{22}\right).$$

References

[1] Binder K and Reger J D 1992 *Adv. Phys.* **41** 547

[2] Scheucher M and Reger J D 1993 *Z. Phys.* B **91** 383

[3] Scheucher M and Reger J D 1992 *Phys. Rev.* B **45** 2499

[4] Singh R R P 1991 *Phys. Rev.* B **43** 6299

[5] Goldschmidt Y Y 1985 *Phys. Rev.* B **31** 4369

[6] Cwilich G and Kirkpatrick T R 1989 *J. Phys. A: Math. Gen.* **22** 4971
Cwilich G 1990 *J. Phys. A: Math. Gen.* **23** 5029

[7] Banavar J and Cieplak M 1989 *Phys. Rev.* B **39** 9633

[8] Schreider G and Reger J D companion paper

[9] Singh R R P and Chakravarty S 1987 *Phys. Rev.* B **36** 546